Tutorial 3

Throughout, V is a finite-dimensional inner product space.

1. Let U_1 and U_2 be subspaces of V. Show that

$$(U_1 + U_2)^\perp = U_1^\perp \cap U_2^\perp$$

2. Let $T \in \mathcal{L}(V)$ and suppose $\mathbb{F} = \mathbb{C}$. Show there exists unique $A, B \in \mathcal{L}(V)$ such that A and B are self-adjoint and $T = A + iB$.

3. Let $T \in \mathcal{L}(V)$ be such that $T^*T = 0$. What can you conclude about T?

4. Let $V = \mathcal{P}_1(\mathbb{R})$ be an inner product space with inner product

$$\langle f, g \rangle = f(0)g(0) + f(1)g(1)$$

Define $T \in \mathcal{L}(V)$ as

$$T(a + bx) = bx$$

   (a) Find T^* explicitly and conclude that T is not self-adjoint.

   (b) Let A be the matrix of T with respect to the standard basis $\{1, x\}$. Show that the conjugate transpose of A is equal to A even though T is not self-adjoint. How is this not a contradiction?

5. Let U be a subspace of V.

   (a) Show that P_U is self-adjoint.

   (b) Show that P_U is diagonalizable.

6. Let $T \in \mathcal{L}(V)$.

   (a) Show that $\dim \operatorname{null} T^* = \dim \operatorname{null} T$ and $\dim \operatorname{range} T^* = \dim \operatorname{range} T$.

   (b) Suppose $T^*T = TT^*$. Show that $\operatorname{null} T^* = \operatorname{null} T$ and $\operatorname{range} T^* = \operatorname{range} T$.

7. In this problem we will establish how to construct the least squares approximating line of a given collection of data points.

Suppose we have points $(x_0, y_0), \ldots, (x_n, y_n)$ in $\mathbb{R}^2$ with the x_i all distinct. If there exists a line $y = mx + b$ going through these points then $y_0 = mx_0 + b$, $y_1 = mx_1 + b$, and so on. In other words,

$$Av - y = 0$$

where

$$A = \begin{pmatrix} x_0 & 1 \\ x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{pmatrix}, \qquad v = \begin{pmatrix} m \\ b \end{pmatrix}, \qquad y = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}$$

As such, asking whether there exists a line passing through all the above data points is equivalent to asking whether there exists $v \in \mathbb{R}^2$ such that $Av - y = 0$.

Of course, there almost never exists a line that goes through all the data points. We define a *least squares approximating line* as a line that makes $\|Av - y\|$ (relative to the dot product) as small as possible. Recall that if we fix $v \in V$ and a subspace $U \subseteq V$, then $\|v - u\|$ subject to the condition $u \in U$ is minimal when $u = P_U(v)$.

(a) With the above in mind, find an inner product space V, a subspace U, and an element $v \in V$ such that $u = P_U(v)$ is the least squares approximating line for the data points $(x_0, y_0), \ldots, (x_n, y_n)$.

(b) Show that $y = 2x + \frac{5}{3}$ is the least squares approximating line for the data points $(-1, 0)$, $(0, 1)$, and $(1, 4)$.